Practice *u*-substitution foundation skill:

Do: $\int \cos \alpha x \, dx$ where α is a constant

$u = \alpha x$

$du = \alpha \, dx \Rightarrow \frac{1}{\alpha} du = dx$

$= \frac{1}{\alpha} \int \cos u \, du$

$= \boxed{\frac{1}{\alpha} \sin \alpha x + C}$

Do: $\int e^{\beta x} dx$ where β is a constant

$u = \beta x$

$\frac{1}{\beta} du = dx$

$= \frac{1}{\beta} \int e^u \, du$

$= \frac{1}{\beta} e^u + C$

$= \boxed{\frac{1}{\beta} e^{\beta x} + C}$

power function simplifies when take derivative

exponential function does not simplify as a derivative

$(x^3)' = (3x^2)' = (6x)' = 6' = 0$

$(e^x)' = (e^x)' = e^x$

Integration by Parts (IBP)

one function will becomes ***u*** and the other function becomes ***dv***.

$$\boxed{\int u \, dv = uv - \int v \, du}$$

ex. $\int \underbrace{\overset{u}{x} e^{3x} dx}_{dv}$

$u = x$ $\quad dv = e^{3x} dx$

$du = dx$ $\quad v = \frac{1}{3} e^{3x}$

$\int u \, dv = uv - \int v \, du$

$= \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx$

$= \frac{1}{3} x e^{3x} - \frac{1}{3} \cdot \frac{1}{3} e^{3x} + C = \boxed{\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C}$

Do the following *two* problems, which may or may not require IBP.

Do: $\int 3x \cos x \, dx$ $= 3 \int \overset{u}{x} \overset{dv}{\cos x \, dx} = uv - \int v \, du$

$u = x$ $\quad dv = \cos x \, dx$

$du = dx$ $\quad v = \sin x$

$= 3\left(x \sin x - \int \sin x \, dx\right)$

$= 3\left(x \sin x + \cos x + C\right)$

$= 3x \sin x + 3\cos x + \text{"}3C\text{"}$

C

$(\cos x)' = -\sin x$

Do: $\int \sec^2\left(\frac{x}{\pi}\right) dx$ u-sub:

$u = \frac{1}{\pi} \cdot x$

$du = \frac{1}{\pi} dx \Rightarrow \pi \, du = dx$

HINT: $(\tan x)' = \sec^2 x$

$= \pi \int \sec^2 u \, du$

$= \pi \cdot \tan u + C$

$= \boxed{\pi \tan\left(\frac{x}{\pi}\right) + C}$

Recall: Sometimes it's necessary to use IBP multiple times:

ex. $\int 4t^2 e^{6t}\, dt = 4\int \underbrace{t^2}_{u} \underbrace{e^{6t}\, dt}_{dv}$

$u = t^2 \qquad dv = e^{6t}\,dt$

$du = 2t\,dt \qquad v = \frac{1}{6}e^{6t}$

$= uv - \int v\, du$

$= 4\left(\frac{1}{6}t^2 e^{6t} - 2\cdot\frac{1}{6}\int \underbrace{t}_{u}\, \underbrace{e^{6t}\, dt}_{dv}\right)$

$u = t \qquad dv = e^{6t}\,dt$

$du = dt \qquad v = \frac{1}{6}e^{6t}$

$uv - \int v\,du$

$= 4\left(\frac{1}{6}t^2 e^{6t} - \frac{1}{3}\left(\frac{1}{6}te^{6t} - \frac{1}{6}\int e^{6t}\,dt\right)\right)$

$= 4\left(\frac{1}{6}t^2 e^{6t} - \frac{1}{3}\left(\frac{1}{6}te^{6t} - \frac{1}{6}\cdot\frac{1}{6}e^{6t}\right)\right) + C$

$36 \times 3 = 108$

$= \frac{4}{6}t^2 e^{6t} - \frac{4}{18}te^{6t} + \frac{4}{108}e^{6t} + C$

$= \boxed{\frac{2}{3}t^2 e^{6t} - \frac{2}{9}te^{6t} + \frac{1}{27}e^{6t} + C}$

Do: $\int 7x^2 \cdot e^{5x}\, dx$

$u = x^2 \qquad dv = e^{5x}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{5}e^{5x}$

$= 7\int x^2 e^{5x}\, dx$

$= 7\left(\frac{1}{5}x^2 e^{5x} - 2\cdot\frac{1}{5}\int xe^{5x}\,dx\right)$

$u = x \qquad dv = e^{5x}\,dx$

$du = dx \qquad v = \frac{1}{5}e^{5x}$

$= 7\left(\frac{1}{5}x^2 e^{5x} - \frac{2}{5}\left(\frac{1}{5}xe^{5x} - \frac{1}{5}\int e^{5x}\,dx\right)\right)$

$= 7\left(\frac{1}{5}x^2 e^{5x} - \frac{2}{25}xe^{5x} + \frac{2}{25}\cdot\frac{1}{5}e^{5x}\right) + C$

$= \boxed{\frac{7}{5}x^2 e^{5x} - \frac{14}{25}xe^{5x} + \frac{14}{125}e^{5x} + C}$

Last Time, it was more advantageous to let *u* = ln *x*: $\int x^5 \ln x \, dx$

$u = \ln x \qquad dv = x^5 \, dx$

$\underbrace{du = \frac{1}{x} dx} \qquad v = \frac{x^6}{6}$

$= \int \ln x \cdot x^5 \, dx$

$= uv - \int v \, du$

$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^6 \cdot \frac{1}{x} \, dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^5 \, dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \frac{x^6}{6} + C$

$= \frac{1}{6} x^6 \ln x - \frac{1}{36} x^6 + C$

The next one is similar but notice it's a **definite** integral:

ex. $\int_1^2 \frac{\ln x}{x^2} dx = \int_1^2 \ln x \cdot \frac{1}{x^2} dx$

$u = \ln x \qquad dv = \frac{1}{x^2} dx = x^{-2} dx$

$\underbrace{du = \frac{1}{x} dx} \qquad v = -x^{-1} = -\frac{1}{x}$

$u \quad v \qquad \int v\,du$

$= \ln x \left(-\frac{1}{x}\right)\Big|_1^2 + \int_1^2 \frac{1}{x} \cdot \underbrace{\frac{1}{x} dx}$

$\frac{1}{x} \cdot \frac{1}{x} = \frac{1}{x^2} = x^{-2}$

$= -\frac{\ln x}{x}\Big|_1^2 + \int_1^2 x^{-2} \, dx$

$= -\frac{\ln x}{x}\Big|_1^2 - \frac{1}{x}\Big|_1^2$

$F(b) - F(a)$

$= -\left(\frac{\ln 2}{2} - \cancel{\ln 1}^{\,0}\right) - \left(\frac{1}{2} - 1\right)$

$= -\frac{\ln 2}{2} - \left(-\frac{1}{2}\right) = \boxed{-\frac{\ln 2}{2} + \frac{1}{2}}$

Like *u*-sub, sometimes the "second" function isn't obvious:

ex. $\int_0^1 \underbrace{\arctan x}_{u}\, \underbrace{dx}_{dv}$

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x$

$= uv\Big|_0^1 - \int_0^1 v\,du$

$= x\arctan x\Big|_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2}\,dx \longleftarrow$ use u-sub

$u = 1+x^2$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$u_a = 1+0 = 1$

$u_b = 1+1^2 = 2$

$= x\arctan x\Big|_0^1 - \frac{1}{2}\int_1^2 \frac{1}{u}\,du$

$= \arctan 1 - 0 - \frac{1}{2}\ln|u|\Big|_1^2$

$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \cancel{\ln 1}^{\,0})$

$= \boxed{\frac{\pi}{4} - \frac{\ln 2}{2}}$

$\tan\frac{\pi}{4} = 1 \quad \therefore \arctan 1 = \frac{\pi}{4}$

How to decide which is *u* and which is *dv*?

u

L logarithms

I inverse (trig)

A algebraic (power)

T trigonometric

E exponential

dv

rule of thumb: <u>not</u> steadfast

Then … there's this …

$\int e^x \sin x \, dx$

LIATE

$u = e^x$ $\quad dv = \sin x \, dx$

$du = e^x dx$ $\quad v = -\cos x$

$$\int e^x \sin x \, dx = -\cos x \, e^x + \underbrace{\int e^x \cos x \, dx}_{\text{IBP}}$$

$u = e^x$ $\quad dv = \cos x \, dx$

$du = e^x dx$ $\quad v = \sin x$

$$\underbrace{\int e^x \sin x \, dx}_{\text{LIKE TERMS}} = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x \, dx}_{\text{LIKE TERMS}}$$

$$+\int e^x \sin x \, dx \qquad\qquad +\int e^x \sin x \, dx$$

← cancels

$$\frac{2\int e^x \sin x \, dx}{2} = \frac{-e^x \cos x + e^x \sin x}{2}$$

$$\int e^x \sin x \, dx = \boxed{\frac{1}{2}\left(e^x \sin x - e^x \cos x\right)}$$

Try on your own: work through problem above with *u* = sin*x*